# An Internal Differential Calculus to The Simply-Typed Lambda Calculus

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**Abstract**: Introduces a representation of banach spaces, differential maps, and linear operators based on coherence spaces and stable / linear maps. Moreover, introduces pseudo-differential stable maps to coherence spaces which induces a calculus of derivatives in the usual, analytic sense internal to the simply-typed lambda calculus.

## Contents



## Section 0 Preliminary

### 0.1 Order Theory

Much of the following text is built upon order theory. A refresher on definitions.

**DEF** Let $(P, \leq)$ be a poset and $D \subseteq P$ a subset. Then

1. $D$ is a *directed set* iff $D \neq \emptyset$ and $\forall x, y \in D.\, \exists z \in D$ such that $x \leq z$ and $y \leq z$,
2. $D$ is a *bounded set* iff $D \neq \emptyset$ and $\exists z \in P$ such that $\forall x \in D.\; x \leq z$ (called an *upper bound* of $D$), and
3. $D$ has a *supremum*, denoted $\sup D$, iff it has a least upper bound.

**DEF** Let $(P, \leq)$ be a poset. An element $c \in P$ is *compact* iff $\forall D \subseteq P$ directed with supremum, if $c \leq \sup D$ then $\exists x \in D$ such that $c \leq x$. The set of compact elements in $P$ is denoted $K(P)$.

For clarity, the union of a directed set $D$ is denoted $\bigcup^{\uparrow} D$. A particularly relevant structure is the following.

**DEF** A poset $(P, \leq)$ is a *Scott domain* iff

1. $P$ is *directed complete*, i.e. all directed subsets in $P$ have a supremum,
2. $P$ is *bounded complete*, i.e. all bounded subsets in $P$ have a supremum, and
3. $P$ is *algebraic*, i.e. $\forall x \in P.\, \exists D \subseteq K(P)$ that is directed such that $x = \sup D$.

## Section 1 Coherence Spaces

Denotational semantics aims to model $\lambda$-calculi by interpreting reductions as equality. Dana Scott introduced Scott topologies which interprets types as topological spaces and arrows as continuous maps. However, this required order-theoretic constructions far removed from the geometric spirit of topology.

### 1.1 Definition

Jean-Yves Girard took Scott's insights to build the following (more faithful) model.

**DEF** A *coherence space* is a nonempty set (of sets) $\mathbf{X}$ which satisfies

1. down closure: $\forall \alpha \in \mathbf{X}$ if $\alpha' \subseteq \alpha$ then $\alpha' \in \mathbf{X}$, and
2. binary completeness: $\forall M \subseteq \mathbf{X}$ if $\forall \alpha, \alpha' \in M.\ \alpha \cup \alpha' \in \mathbf{X}$ then $\bigcup M \in \mathbf{X}$.

Coherence spaces are partially ordered by inclusion $\subseteq$. Moreover, each has the *undefined object*, $\emptyset \in \mathbf{X}$. We may also treat coherence spaces as undirected graphs.

**DEF** Let $\mathbf{X}$ be a coherence space. The set of *tokens* in $\mathbf{X}$ is $|\mathbf{X}| \coloneqq \bigcup \mathbf{X}$. Two tokens $x, x' \in |\mathbf{X}|$ are *coherent* iff $\{x, x'\} \in \mathbf{X}$, denoted $x \frown\!\!\!\!\smile x'$. They are *strictly coherent* iff $x \neq x' \wedge x \frown\!\!\!\!\smile x'$, denoted $x \frown x'$.

Coherence is a reflexive symmetric relation mutually definable by strict coherence. Thus $|\mathbf{X}|$ equipped with $\frown$ defines a graph, called the *web* of $\mathbf{X}$. In fact from a web, we can recover the corresponding coherence space by

$$\alpha \in \mathbf{X} \iff \alpha \subseteq |\mathbf{X}| \wedge \forall x, x' \in \alpha.\ x \frown\!\!\!\!\smile x'.$$

Thus $\mathbf{X}$ is exactly the set of complete subgraphs in the web of $\mathbf{X}$. In light of this bijection, we could have equivalently defined coherence spaces in terms of undirected graphs $(|\mathbf{X}|, \frown)$. In the terminology of Graph Theory,

**DEF** Let $\mathbf{X}$ be a coherence space. A *clique* in $\mathbf{X}$ is an element of $\mathbf{X}$. Two cliques $\alpha, \alpha' \in \mathbf{X}$ are *compatible* iff $\alpha \cup \alpha' \in \mathbf{X}$, denoted $\alpha \frown\!\!\!\!\smile \alpha'$. They are *strictly compatible* iff $\alpha \neq \alpha' \wedge \alpha \frown\!\!\!\!\smile \alpha'$, denoted $\alpha \frown \alpha'$.

For convenience, $\mathbf{X}_{\text{fin}}$ and $\mathbf{X}_{\text{max}}$ denote the set of *finite cliques* and *maximal cliques* in $\mathbf{X}$ respectively (which are nonempty). The aim is to interpret a type as a coherence space $\mathbf{X}$ and terms of this type as cliques in $\mathbf{X}$. To effectively treat terms as cliques, we need to introduce a notion of finite approximation.

**DEF** Let $\mathbf{X}$ be a coherence space. An *approximant* of $\alpha \in \mathbf{X}$ is a subclique $\alpha' \subseteq \alpha$.

If we consider terms as total objects then approximants are partial objects. These serve the same purpose that partial maps do for total maps in recursion theory. Finite approximants are especially useful since they represent data we can conceivably attain in finite time with finite resources.

1.1 **PROP** Let $\mathbf{X}$ be a coherence space and $\beta \in \mathbf{X}$. Define $D \coloneqq \{\alpha \subseteq \beta \mid \alpha \in \mathbf{X}_{\text{fin}}\}$. Then $D$ is directed with $\bigcup^{\uparrow} D = \beta$.

*P–f* Suppose $\alpha, \alpha' \in D$. Then $\alpha \cup \alpha' \subseteq \beta$ with $|\alpha \cup \alpha'| < \infty$ and so $\alpha \cup \alpha' \in D$ by down closure. Thus $D$ is directed. Observe $\beta = \bigcup_{x \in \beta} \{x\} \subseteq \bigcup^{\uparrow} D \subseteq \beta$. Thus $\bigcup^{\uparrow} D = \beta$. ∎

### 1.2 Domain Theory

Since coherence spaces are derived from Scott's insights, it is natural to ask whether Scott's analyses carry over.

1.2 **LEM** Let $\mathbf{X}$ be a coherence space and $D \subseteq \mathbf{X}$ directed. Then $\sup D = \bigcup^{\uparrow} D \in \mathbf{X}$.

*P–f* Let $D \subseteq \mathbf{X}$ be directed. Suppose $\alpha, \alpha' \in D$. By definition, $\exists \beta \in D$ such that $\alpha \cup \alpha' \subseteq \beta$. Since $\beta \in \mathbf{X}$, $\alpha \cup \alpha' \in \mathbf{X}$ by down closure. Thus $\bigcup^{\uparrow} D \in \mathbf{X}$ by binary completeness and hence an upper bound of $D$. Suppose $\beta' \in \mathbf{X}$ is an upper bound of $D$. Then $\forall \alpha \in D.\ \alpha \subseteq \beta'$, i.e. $\bigcup^{\uparrow} D \subseteq \beta'$. Therefore $\sup D = \bigcup^{\uparrow} D$. ∎

1.3 **LEM** Let $\mathbf{X}$ be a coherence space. Then $\mathbf{X}_{\text{fin}} = K(\mathbf{X})$, the set of compact elements in $\mathbf{X}$.

*P–f* ($\Rightarrow$) Let $c = \{x_1, \ldots, x_n\} \in \mathbf{X}_{\text{fin}}$. Suppose $D \subseteq \mathbf{X}$ is directed with $c \subseteq \sup D$. Then $c \subseteq \bigcup^{\uparrow} D$ by Lemma 1.2 and so $\forall x_k \in c.\ \exists \alpha_k \in D$ such that $x_k \in \alpha_k$. Thus $c \subseteq \bigcup_k \alpha_k$. By definition of $D$, $\exists \beta \in D$ such that $\bigcup_k \alpha_k \subseteq \beta$ and hence $c \subseteq \beta$. Therefore $c \in K(\mathbf{X})$.

($\Leftarrow$) Let $c \in K(\mathbf{X})$. Define $D \coloneqq \{\alpha \subseteq c \mid \alpha \in \mathbf{X}_{\text{fin}}\}$. Then $D$ is directed with $c = \bigcup^{\uparrow} D$ by Proposition 1.1. Observe $c = \sup D$ by proof of Lemma 1.2. By compactness, $\exists \beta \in D$ such that $c \subseteq \beta$. Therefore $c \in \mathbf{X}_{\text{fin}}$. ∎

These properties of coherence spaces culminate to the relevant order-theoretic structure.

1.4 **THM** Let $\mathbf{X}$ be a coherence space. Then $(\mathbf{X}, \subseteq)$ is a Scott domain.

*P–f*
1. $\mathbf{X}$ is directed complete by Lemma 1.2.
2. Let $D \subseteq \mathbf{X}$ be bounded with upper bound $\beta \in \mathbf{X}$. Suppose $\alpha, \alpha' \in D$. By definition, $\alpha \cup \alpha' \subseteq \beta$ and so $\alpha \cup \alpha' \in \mathbf{X}$ by down closure. Thus $\bigcup D \in \mathbf{X}$ by binary completeness and hence an upper bound of $D$. Suppose $\beta' \in \mathbf{X}$ is an upper bound of $D$. Then $\forall \alpha \in D.\ \alpha \subseteq \beta'$, i.e. $\bigcup D \subseteq \beta'$. Therefore $\sup D = \bigcup D$.
3. Let $\beta \in \mathbf{X}$. Define $D \coloneqq \{\alpha \subseteq \beta \mid \alpha \in \mathbf{X}_{\text{fin}}\}$. Note $D \subseteq K(\mathbf{X})$ by Lemma 1.3. Moreover, $D$ is directed with $\beta = \bigcup^{\uparrow} D$ by Proposition 1.1. Therefore $\beta = \sup D$ by Lemma 1.2. ∎

## 1.3 Topology

Since coherence spaces are Scott domains, they also admit a Scott topology.

1.5 **THM** Let $\mathbf{X}$ be a coherence space. Define $\mathcal{B} \coloneqq \{\uparrow\alpha \mid \alpha \in \mathbf{X}_{\text{fin}}\}$ where $\uparrow\alpha \coloneqq \{\beta \in \mathbf{X} \mid \alpha \subseteq \beta\}$. Then $\mathcal{B}$ is a basis for the topology $\tau$ on $\mathbf{X}$ called the *Scott topology*.

*P–f*
1. Observe $\mathbf{X} = \uparrow\emptyset \in \mathcal{B}$. Therefore $\mathbf{X} = \bigcup \mathcal{B}$.
2. Let $\uparrow\alpha, \uparrow\alpha' \in \mathcal{B}$. Suppose $\beta \in \uparrow\alpha \cap \uparrow\alpha'$. Then $\alpha \cup \alpha' \subseteq \beta$ with $|\alpha \cup \alpha'| < \infty$ and so $\alpha \cup \alpha' \in \mathbf{X}_{\text{fin}}$ by down closure. Thus $\beta \in \uparrow(\alpha \cup \alpha') \in \mathcal{B}$. Therefore $\beta \in \uparrow(\alpha \cup \alpha') \subseteq \uparrow\alpha \cap \uparrow\alpha'$ since $\alpha, \alpha' \subseteq \alpha \cup \alpha'$. ∎

For convenience, if $x \in |\mathbf{X}|$ then the basis element $\uparrow\{x\}$ is denoted $\uparrow x$. Note $U \subseteq \mathbf{X}$ is an open subset under the Scott topology, called *Scott-open*, iff it is an upper set and is innaccessible by directed unions, i.e. every directed set $D \subseteq \mathbf{X}$ with $\sup D \in U$ have $D \cap U \neq \emptyset$. Some notable properties.

1.5.1 **COR** Let $\mathbf{X}$ be a coherence space. Then $\mathbf{X}$ is a $T_0$-space.

*P–f* Let $\alpha, \alpha' \in \mathbf{X}$ be distinct. Then $\alpha \setminus \alpha' \neq \emptyset$ or $\alpha' \setminus \alpha \neq \emptyset$. WLOG suppose $\exists x \in \alpha \setminus \alpha'$. Observe $\alpha \in \uparrow x \in \tau$ but $\alpha' \notin \uparrow x$. Therefore $\exists U \in \tau$ such that $\alpha \in U$ and $\alpha' \notin U$ or $\alpha \notin U$ and $\alpha' \in U$. ∎

1.5.2 **COR** Let $\mathbf{X}$ be a coherence space with $|\mathbf{X}|$ countable. Then $\mathbf{X}$ is countably based.

*P–f* Observe $\mathbf{X} \subseteq \mathcal{P}(|\mathbf{X}|)$ and thus $\mathbf{X}_{\text{fin}} = \{\alpha \in \mathbf{X} \mid |\alpha| < \infty\} \subseteq \{\alpha \in \mathcal{P}(|\mathbf{X}|) \mid |\alpha| < \infty\}$. Since the set of finite subsets of a countable set is countable, $\mathbf{X}_{\text{fin}}$ is countable. Therefore $\mathcal{B}$ is countable. ∎

Note a map between coherence spaces $F : \mathbf{X} \to \mathbf{Y}$ is continuous under the Scott topology, called *Scott-continuous*, iff $F$ preserves directed unions.

## 1.4 Stable Maps

Ideally an arrow term in a $\lambda$-calculus should be interpreted as a map between coherence spaces. We seek to only consider maps that are, in some sense, computable. Gérard Berry proposed stable maps where finite parts of the output can be computed by a least approximant of the input. Formally.

**DEF** Let $F : \mathbf{X} \to \mathbf{Y}$ be a map between coherence spaces. A pair $(\alpha, y) \in \mathbf{X}_{\text{fin}} \times |\mathbf{Y}|$ is a *minimal pair* of $F$ iff $y \in F(\alpha)$ and $\neg\exists \alpha' \subset \alpha$ such that $y \in F(\alpha')$. The *trace* of $F$ is the set of minimal pairs of $F$, denoted $\text{tr}(F)$.

**DEF** Let $\mathbf{X}, \mathbf{Y}$ be coherence spaces. A map $F$ from $\mathbf{X}$ to $\mathbf{Y}$ is *stable*, denoted $F : \mathbf{X} \to_{\text{st}} \mathbf{Y}$, iff it satisfies

1. monotonicity, i.e. if $\alpha \subseteq \beta \in \mathbf{X}$ then $F(\alpha) \subseteq F(\beta)$, and
2. stability, i.e. if $y \in F(\alpha)$ then $\exists!\ \alpha_0 \subseteq \alpha$ such that $(\alpha_0, y) \in \text{tr}(F)$.

Berry proposed stable maps as a model of sequential computation, i.e. that outputs are "constructed" from the input. Indeed, monotonicity asserts approximants are preserved and stability asserts outputs are deterministic. In fact from a trace $\text{tr}(F) \subseteq \mathbf{X}_{\text{fin}} \times |\mathbf{Y}|$, we can recover the corresponding stable map $F$ by

$$y \in F(\alpha) \iff \exists \alpha_0 \subseteq \alpha.\ (\alpha_0, y) \in \text{tr}(F).$$

This not only proves that the traces of stable maps are unique, but also further supports the idea that stable maps model sequential computation. Besides the direct computational formulation, there is an equivalent algebraic formulation of stability independently discovered by Girard.

1.6 **PROP** Let $F : \mathbf{X} \to \mathbf{Y}$ be a map between coherence spaces. Then $F$ is stable iff it is

1. continuous, i.e. $F(\bigcup^{\uparrow} D) = \bigcup^{\uparrow} F(D)$ where $D \subseteq \mathbf{X}$ is directed, and
2. preserves pullback, i.e. if $\alpha \bigcirc_{\mathbf{X}} \beta$ then $F(\alpha \cap \beta) = F(\alpha) \cap F(\beta)$.

*P–f* (⇒) Let $D \subseteq \mathbf{X}$ be directed. Then $F(D)$ is directed with $\bigcup^{\uparrow} F(D) = \sup F(D) \subseteq F(\bigcup^{\uparrow} D)$ by mono and Lemma 1.2. Suppose $y \in F(\bigcup^{\uparrow} D)$. Then $\exists!\, \alpha_0 \subseteq \bigcup^{\uparrow} D$ such that $(\alpha_0, y) \in \mathrm{tr}(F)$ by stability. Since $|\alpha_0| < \infty$, $\exists \beta \in D$ such that $\alpha_0 \subseteq \beta$ and hence $y \in F(\beta) \subseteq \bigcup^{\uparrow} F(D)$. Thus $F(\bigcup^{\uparrow} D) = \bigcup^{\uparrow} F(D)$.

Let $\alpha \bigcirc_{\mathbf{X}} \beta$. Note $F(\alpha \cap \beta) \subseteq F(\alpha) \cap F(\beta) \subseteq F(\alpha \cup \beta)$ by mono. Suppose $y \in F(\alpha) \cap F(\beta)$. Then $\exists!\, \alpha_0 \subseteq \alpha \cap \beta$ such that $(\alpha_0, y) \in \mathrm{tr}(F)$ by stability. Thus $y \in F(\alpha \cap \beta)$ and so $F(\alpha \cap \beta) = F(\alpha) \cap F(\beta)$. Therefore $F$ is continuous and preserves pullback.

(⇐) Let $\alpha \subseteq \beta \in \mathbf{X}$. Then $F(\alpha) \subseteq F(\alpha) \cup F(\beta) = F(\alpha \cup \beta) = F(\beta)$ by continuity. Thus $F$ is monotone.

Suppose $y \in F(\alpha)$. Then $\alpha \in F^{-1}(\uparrow y)$ is open by continuity. By construction, $\exists \alpha' \in \mathbf{X}_{\text{fin}}$ such that $\alpha \in \uparrow\alpha' \subseteq F^{-1}(\uparrow y)$. Since $|\alpha'| < \infty$, $\exists \alpha_0 \subseteq \alpha' \subseteq \alpha$ such that $(\alpha_0, y) \in \mathrm{tr}(F)$ by well-ordering principle. By pullback, $\alpha_0$ is unique. Thus $F$ satisfies stability. Therefore $F$ is stable. ■

Indeed, this matches our intuition. Continuity asserts that outputs are constructed from finite approximants and pullback asserts that this construction is unique. Girard's criterion tend to be easier to prove, hence its mention. For example, it is evident that the identity map is stable by Proposition 1.6.

1.6.1 **COR** Let $F : \mathbf{X} \to_{\text{st}} \mathbf{Y}$ and $G : \mathbf{Y} \to_{\text{st}} \mathbf{Z}$ be stable maps. Then the composition $G \circ F$ is stable.

*P–f* Note $F$ and $G$ are continuous maps. Thus $G \circ F$ is continuous. Now suppose $\alpha \bigcirc_{\mathbf{X}} \beta$. Then $F(\alpha) \bigcirc_{\mathbf{Y}} F(\beta)$. Thus $[G \circ F](\alpha \cap \beta) = [G \circ F](\alpha) \cap [G \circ F](\beta)$, i.e. $G \circ F$ preserves pullback. Therefore $G \circ F$ is stable. ■

We denote by **Coh** the category of coherence spaces and stable maps. In order to interpret arrow terms as stable maps, we must present the set of stable maps as a coherence space which will model the arrow type.

**DEF** Let $\mathbf{X}, \mathbf{Y}$ be coherence spaces. The *stable exponential* (or internal hom) from $\mathbf{X}$ to $\mathbf{Y}$ is

$$\mathbf{X} \Rightarrow \mathbf{Y} \coloneqq \{\mathrm{tr}(F) \mid F : \mathbf{X} \to_{\text{st}} \mathbf{Y}\}.$$

This is a coherence space where $|\mathbf{X} \Rightarrow \mathbf{Y}| = \mathbf{X}_{\text{fin}} \times |\mathbf{Y}|$ and $(\alpha, y) \frown (\alpha', y')$ iff $\alpha \bigcirc_{\mathbf{X}} \alpha'$ implies $y \frown_{\mathbf{Y}} y'$.

For convenience, if $\alpha \in \mathbf{X} \Rightarrow \mathbf{Y}$ then $\widehat{\alpha} : \mathbf{X} \to_{\text{st}} \mathbf{Y}$ denotes the corresponding stable map such that $\alpha = \mathrm{tr}(\widehat{\alpha})$. The bijection between $\mathbf{X} \Rightarrow \mathbf{Y}$ and $\mathbf{X} \to_{\text{st}} \mathbf{Y}$ induces an order relation.

**DEF** Let $\mathbf{X}, \mathbf{Y}$ be coherence spaces. The *Berry order* $\leq_{\mathrm{B}}$ on $\mathbf{X} \to_{\text{st}} \mathbf{Y}$ is given by $F \leq_{\mathrm{B}} G$ iff $\mathrm{tr}(F) \subseteq \mathrm{tr}(G)$.

**TODO** Explore more properties about the berry order that may be helpful to prove stable exponential is indeed an exponential object.

### 1.5 Direct Products

Stable maps generalize to multiple arguments by treating inclusion on the arguments component-wise. The resulting inclusion structure on the arguments behaves like a coherence space. This motivates the following definition.

**DEF** Let $\{\mathbf{X}_j\}_{j\in\Lambda}$ be an indexed family of coherence spaces. The *direct product* (or product) of $\{\mathbf{X}_j\}_{j\in\Lambda}$ is

$$\prod_{j\in\Lambda} \mathbf{X}_j \coloneqq \left\{ \langle \alpha_j \rangle_{j\in\Lambda} \mid \alpha_j \in \mathbf{X}_j \right\} \quad \text{where} \quad \langle \alpha_j \rangle_{j\in\Lambda} \coloneqq \coprod_{j\in\Lambda} \alpha_j$$

This is a coherence space where $|\prod_{j\in\Lambda} \mathbf{X}_j| = \coprod_{j\in\Lambda} |\mathbf{X}_j|$ and $(i, x) \frown (j, x')$ iff $i = j$ implies $x \frown_i x'$.

When clear, we omit the index set $\Lambda$ and denote $\langle \alpha_j \rangle$ as $\vec{\alpha}$. Note $\vec{\alpha} \subseteq \vec{\beta}$ iff $\forall j.\ \alpha_j \subseteq \beta_j$. Also conjunction and disjunction commute with $\langle \cdot \rangle$. Thus a stable map over multiple arguments is equivalent to a stable map over the corresponding direct product. More assuring that this is a suitable product.

1.7 **PROP** Let $\{\mathbf{X}_j\}$ be an indexed family of coherence spaces. Define $\forall k$ the *projection* $\pi_k : \prod_j \mathbf{X}_j \to \mathbf{X}_k$ by $\pi_k(\vec{\alpha}) \coloneqq \alpha_k$. Then $\forall k.\ \pi_k$ is stable.

*P–f* Note if $\vec{\alpha} \subseteq \vec{\beta}$ then $\forall j.\ \alpha_j \subseteq \beta_j$, i.e. $\forall j.\ \pi_j(\vec{\alpha}) \subseteq \pi_j(\vec{\beta})$. Fix $\pi_k$. Let $D \subseteq \prod_j \mathbf{X}_j$ be directed. Then $\pi_k(D)$ is directed by mono with $\bigcup^{\uparrow} D = \langle \bigcup^{\uparrow} \pi_j(D) \rangle$ by construction. Observe

$$\pi_k(\textstyle\bigcup^{\uparrow} D) = \pi_k(\langle \bigcup^{\uparrow} \pi_j(D) \rangle) = \bigcup^{\uparrow} \pi_k(D).$$

Thus $\pi_k$ is continuous. Now let $\vec{\alpha} \frown \vec{\beta}$. Then $\pi_k(\vec{\alpha} \cap \vec{\beta}) = \pi_k(\langle \alpha_j \cap \beta_j \rangle) = \alpha_k \cap \beta_k = \pi_k(\vec{\alpha}) \cap \pi_k(\vec{\beta})$. Thus $\pi_k$ preserves pullback. Therefore $\pi_k$ is stable. ∎

**TODO** Check if the direct product $\prod_i \mathbf{X}_i$ is universal with this property. I.e. let $\mathbf{Q}$ be a coherence space and $f_k : \mathbf{Q} \to_{\text{st}} \mathbf{X}_k$ be stable maps. Prove $\exists!\, f : \mathbf{Q} \to \prod_i \mathbf{X}_i$ such that $\forall k.\ \pi_k \circ f = f_k$.

**TODO** Check if the stable exponential $\mathbf{X} \Rightarrow \mathbf{Y}$ is an exponential object with evaluator $\mathrm{ev} : (\mathbf{X} \Rightarrow \mathbf{Y}) \times \mathbf{X} \to \mathbf{Y}$ defined by $\mathrm{ev}(\mathrm{tr}(f), \alpha) = f(\alpha)$. I.e. let $\mathbf{Q}$ be a coherence space and $g : \mathbf{Q} \times \mathbf{X} \to_{\text{st}} \mathbf{Y}$ be stable maps. Prove ev is stable and $\exists!\, \lambda_g : \mathbf{Q} \to \mathbf{X} \Rightarrow \mathbf{Y}$ such that $\mathrm{ev} \circ (\lambda_g \times \mathrm{id}_{\mathbf{Y}}) = g$.

**TODO** Using the above two results, deduce that the category of coherence spaces **Coh** with stable maps is cartesian closed.

### 1.6 Linear Maps

**TODO** Expand on what linear maps are.

## Section 2 Representations

Computational analysis aims to make computable some abstract mathematical spaces that cannot be dealt with directly by computers. A common approach is the Type-II Theory of Effectivity (TTE) where an abstract space is represented by a partial surjection from some concrete space.

### 2.1 Definition

Kei Matsumoto and Kazushige Terui adapted TTE to represent abstract spaces with coherence spaces.

**DEF** Let $S$ be an arbitrary set and $\mathbf{X}$ a coherence space. A *representation* of $S$ is a partial surjective map $\rho :\subseteq \mathbf{X} \twoheadrightarrow S$, denoted $\mathbf{X} \xrightarrow{\rho} S$ or simply $\rho$. An element $r \in S$ is *realized* by a clique $\alpha \in \mathbf{X}$ via $\rho$ iff $\rho(\alpha) = r$.

Representations allow us to express abstract maps as stable maps. Here we say a diagram commutes iff every path with the same source and target agree on their shared domain of definition.

**DEF** Let $\mathbf{X} \xrightarrow{\rho} S$ and $\mathbf{Y} \xrightarrow{\varphi} T$ be representations. A total map $f : S \to T$ is *realized* by a stable map $F : \mathbf{X} \to_{\text{st}} \mathbf{Y}$ via $\rho$ to $\varphi$ iff $F(\mathrm{dom}(\rho)) \subseteq \mathrm{dom}(\varphi)$ and the following diagram commutes.

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{F} & \mathbf{Y} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\varphi} \\ S & \xrightarrow{f} & T \end{array}$$

Such a map $f$ is *stably realizable* via $\rho$ to $\varphi$.

In fact we can treat stably realizable maps as morphisms between representations. It is evident that the identity map over an abstract space is stably realizable by the stable identity map over its representation.

2.1 **LEM** Let $f : S \to T$ and $g : T \to U$ be stably realizable maps via representations $\mathbf{X} \xrightarrow{\rho} S$, $\mathbf{Y} \xrightarrow{\varphi} T$, and $\mathbf{Z} \xrightarrow{\theta} U$. Then the composition $g \circ f$ is stably realizable via $\rho$ to $\theta$.

*P–f* Let $F : \mathbf{X} \to_{\text{st}} \mathbf{Y}$ and $G : \mathbf{Y} \to_{\text{st}} \mathbf{Z}$ be stable maps that realize $f$ and $g$ respectively. By definition, $f \circ \rho = \varphi \circ F$ on $\mathrm{dom}(\rho)$ and $g \circ \varphi = \theta \circ G$ on $\mathrm{dom}(\varphi)$. Observe $(g \circ f) \circ \rho = g \circ \varphi \circ F = \theta \circ (G \circ F)$ on $\mathrm{dom}(\rho)$ where $G \circ F : \mathbf{X} \to_{\text{st}} \mathbf{Z}$ is stable by Corollary 1.6.1. Therefore $g \circ f$ is stably realizable via $\rho$ to $\theta$. ∎

We denote by $\mathbf{Rep}(\mathbf{Coh})$ the category of representations and stably realizable maps. Many of the constructions on coherence spaces carry over to representations and stably realizable maps.

**DEF** Let $\mathbf{X} \xrightarrow{\rho} S$ and $\mathbf{Y} \xrightarrow{\varphi} T$ be representations. The *product representation* of $\rho$ and $\varphi$ is $\mathbf{X} \times \mathbf{Y} \xrightarrow{\rho\times\varphi} S \times T$ defined by

$$\mathrm{dom}(\rho \times \varphi) := \mathrm{dom}(\rho) \times \mathrm{dom}(\varphi) \quad \text{with} \quad [\rho \times \varphi](\langle \alpha, \alpha' \rangle) := (\rho(\alpha), \varphi(\alpha')) .$$

The *exponential representation* from $\rho$ to $\varphi$ is $\mathbf{X} \Rightarrow \mathbf{Y} \xrightarrow{\rho\to\varphi} \mathcal{SR}(\rho, \varphi)$ defined by

$$\mathrm{dom}(\rho \to \varphi) := \{\alpha \in \mathbf{X} \Rightarrow \mathbf{Y} \mid \exists f_\alpha : S \to T \text{ stably realized by } \widehat{\alpha}\} \quad \text{with} \quad [\rho \to \varphi](\alpha) := f_\alpha$$

where $\mathcal{SR}(\rho, \varphi) \subseteq T^S$ is the set of stably realizable maps via $\rho$ to $\varphi$.

**TODO** Prove the category $\mathbf{Rep}(\mathbf{Coh})$ of representations with stably realizable maps is cartesian closed. Also it is regular and locally cartesian closed.

### 2.2 Spanned Representations

In general, an abstract space may have many representations with wildly varying properties. In TTE, admissibility gives a criterion for computationally reasonable representations. Matsumoto and Terui adapt this concept to coherence spaces with slight modifications due to the stronger restrictions of stable maps over continuous maps.

**DEF** Let $\mathbf{X} \xrightarrow{\rho} S$ be a representation. A *spanning forest* for $\rho$ is a set $\mathcal{F} \subseteq \mathbf{X}_{\mathrm{fin}}$ such that

1. $(\mathcal{F}, \subseteq)$ is a forest, in particular $\forall \alpha, \alpha' \in \mathcal{F}$ if $\alpha \frown \alpha'$ then $\alpha \subseteq \alpha'$ or $\alpha' \subseteq \alpha$, and
2. $\mathcal{F}$ spans $\mathrm{dom}(\rho)$, i.e. $\forall \alpha \in \mathbf{X}.\ \alpha \in \mathrm{dom}(\rho)$ iff $\exists$ a maximal chain $\{\alpha_j\} \subseteq \mathcal{F}$ such that $\alpha = \bigcup_j \alpha_j$.

If such an $\mathcal{F}$ exists then $\rho$ is a *spanned representation*.

We denote by $\mathbf{SpnRep}(\mathbf{Coh})$ the full subcategory of $\mathbf{Rep}(\mathbf{Coh})$ that consists of spanned representations. Spanning forests draw inspiration from the prefix tree $(\{0,1\}^\star, \sqsubseteq)$ which, in some sense, spans the Cantor space $\{0,1\}^\omega$. Indeed the naïve representation of the Cantor space is a spanned representation.

2.2 *Ex* Let $\mathbf{C}$ be the coherence space with $|\mathbf{C}| = \{0,1\}^\star$ and $u \frown w$ iff $u \sqsubseteq w$ or $w \sqsubseteq u$. Then $\mathbf{C}_{\max} = \{\downarrow w \mid w \in \{0,1\}^\omega\}$ where $\downarrow w := \{u \in \{0,1\}^\star \mid u \sqsubseteq w\}$ is unique $\forall w$. Thus the map $\rho_{\mathbf{C}} :\subseteq \mathbf{C} \twoheadrightarrow \{0,1\}^\omega$ defined by

$$\mathrm{dom}(\rho_{\mathbf{C}}) := \mathbf{C}_{\max} \quad \text{with} \quad \rho_{\mathbf{C}}(\downarrow w) := w$$

is a representation of the Cantor space. Also $\mathcal{F}_{\mathbf{C}} := \{\downarrow w \mid w \in \{0,1\}^*\} \subseteq \mathbf{C}_{\mathrm{fin}}$ is a spanning forest for $\rho_{\mathbf{C}}$. Therefore $\rho_{\mathbf{C}}$ is a spanned representation.

Unlike with general representations, spanning forests give guarantees on the domain of spanned representations that we will exploit later on. Fortunately this is not limiting.

2.3 **LEM** Let $\mathbf{X} \xrightarrow{\rho} S$ and $\mathbf{Y} \xrightarrow{\varphi} T$ be spanned representations. Then $\mathbf{X} \times \mathbf{Y} \xrightarrow{\rho\times\varphi} S \times T$ is a spanned representation.

*P–f* Let $\mathcal{F}_\rho$ and $\mathcal{F}_\varphi$ be spanning forests for $\rho$ and $\varphi$ respectively. Note every $\alpha \in \mathrm{dom}(\rho)$ is either a leaf or the limit of an infinite path in $\mathcal{F}_\rho$. Define $h : \mathcal{F}_\rho \to \mathbb{N}$ by $h(\alpha) := \left|\{\alpha' \subseteq \alpha \mid \alpha' \in \mathcal{F}_\rho\}\right|$ given that $\mathcal{F}_\rho \subseteq \mathbf{X}_{\mathrm{fin}}$. Define similarly for $\mathcal{F}_\varphi$. Now define $\mathcal{F}_{\rho\times\varphi} \subseteq (\mathbf{X} \times \mathbf{Y})_{\mathrm{fin}}$ by

$$\langle \alpha, \beta \rangle \in \mathcal{F}_{\rho\times\varphi} \iff \begin{cases} h(\alpha) = h(\beta), \text{ or} \\ \alpha \text{ is a leaf and } h(\alpha) < h(\beta), \text{ or} \\ \beta \text{ is a leaf and } h(\alpha) > h(\beta). \end{cases}$$

where $\alpha \in \mathcal{F}_\rho$ and $\beta \in \mathcal{F}_\varphi$. By inspection, $\mathcal{F}_{\rho\times\varphi}$ spans $\mathrm{dom}(\rho \times \varphi)$ and $\vec{\alpha}, \vec{\alpha}' \in \mathcal{F}$ are comparable iff $\vec{\alpha} \frown \vec{\alpha}'$. Thus $\mathcal{F}_{\rho\times\varphi}$ is a spanning forest. Therefore the product representation is a spanned representation. ∎

**TODO** Check if the exponential representation is a spanned representation as well.

Note $\forall \alpha \in \mathrm{dom}(\rho).\ \exists!$ maximal chain in $\mathcal{F}$ that approximates $\alpha$. Conversely $\forall \alpha \in \mathcal{F}.\ \uparrow\alpha \cap \mathrm{dom}(\rho) \neq \emptyset$. Moreover every maximal chain must be countable. Thus $\mathrm{dom}(\rho)$ is a pairwise incompatible set of countable sets.

2.4 **LEM** Let $\mathbf{X} \xrightarrow{\rho} S$ be a spanned representation with spanning forest $\mathcal{F}$. Let $\langle\beta_n\rangle \in \mathrm{dom}(\rho)^\omega$ be a sequence and $\{\alpha_j\} \subseteq \mathcal{F}$ a maximal chain. Then $\beta_n \to \bigcup_j \alpha_j \in \mathrm{dom}(\rho)$ iff $\forall N.\ \langle\beta_n\rangle$ is eventually in $\uparrow\alpha_N \cap \mathrm{dom}(\rho)$.

**TODO** Typeset proof.

2.5 **THM** Let $\mathbf{X} \xrightarrow{\rho} S$ be a spanned representation. Then $\mathrm{dom}(\rho)$ is a sequential subspace.

**TODO** Typeset proof.

### 2.3 Admissibility

Matsumoto demonstrated that $\mathbf{SpnRep}(\mathbf{Coh})$ is categorically equivalent[1] to the category $\mathbf{Rep}(\mathbb{B})$ of TTE representations. Thus the concept of admissibility naturally translates. Here we say a partial map is continuous iff it is continuous on its domain as a subspace.

**DEF** Let $\mathbb{Y}$ be a topological space. A continuous spanned representation $\mathbf{Y} \xrightarrow{\varphi} \mathbb{Y}$ is *admissible* iff $\forall$ continuous spanned representations $\mathbf{X} \xrightarrow{\rho} \mathbb{Y}_0$ (given subspace $\mathbb{Y}_0 \subseteq \mathbb{Y}$) the inclusion map $i : \mathbb{Y}_0 \hookrightarrow \mathbb{Y}$ is stably realizable via $\rho$ to $\varphi$.

Note admissible representations on fixed topological spaces are interchangeable as follows.

2.6 **PROP** Let $\mathbf{X}_k \xrightarrow{\rho_k} \mathbb{X}$ and $\mathbf{Y}_k \xrightarrow{\varphi_k} \mathbb{Y}$ be admissible representations for $k = 1, 2$. Then $\mathcal{SR}(\rho_1, \varphi_1) = \mathcal{SR}(\rho_2, \varphi_2)$.

*P–f* Note it suffice to prove $\mathcal{SR}(\rho_1, \varphi_1) \subseteq \mathcal{SR}(\rho_2, \varphi_2)$ by symmetry. Suppose a map $f : \mathbb{X} \to \mathbb{Y}$ is stably realizable via $\rho_1$ to $\varphi_1$. Then the following diagram commutes by admissibility.

$$\begin{array}{ccccccc} \mathbf{X}_2 & \overset{G_{\mathbf{X}}}{\dashrightarrow} & \mathbf{X}_1 & \overset{F}{\dashrightarrow} & \mathbf{Y}_1 & \overset{G_{\mathbf{Y}}}{\dashrightarrow} & \mathbf{Y}_2 \\ {\scriptstyle \rho_2}\downarrow & & {\scriptstyle\rho_1}\downarrow & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} \\ \mathbb{X} & \overset{i_{\mathbb{X}}}{\hookrightarrow} & \mathbb{X} & \xrightarrow{f} & \mathbb{Y} & \overset{i_{\mathbb{Y}}}{\hookrightarrow} & \mathbb{Y}\end{array}$$

where $G_{\mathbf{X}}$, $G_{\mathbf{Y}}$, and $F$ are stable maps that realize $i_{\mathbb{X}}$, $i_{\mathbb{Y}}$, and $f$ respectively. Therefore $f = i_{\mathbb{Y}} \circ f \circ i_{\mathbb{X}}$ is realized by the stable map $G_{\mathbf{Y}} \circ F \circ G_{\mathbf{X}} : \mathbf{X}_2 \to_{\mathrm{st}} \mathbf{Y}_2$, i.e. $f$ is stably realizable via $\rho_2$ to $\varphi_2$. ∎

Hence stable realizability is independent of choice of admissible representation. In fact stable realizability coincides with sequential continuity. First a technical construction.

Consider the coherence space $\mathbf{C}$ in Example 2.2 representing $\{0,1\}^\omega$. Define the subspace $\mathbf{C}_0 := \{\alpha_n \mid n \in \mathbb{N} \cup \{\infty\}\}$ where $\forall n \in \mathbb{N}.\ \alpha_n := \downarrow 0^n 1^\omega$ and $\alpha_\infty := \downarrow 0^\omega$. Then $\alpha_n \to \alpha_\infty \in \mathbf{C}_0$ (and in $\mathbf{C}$).

2.7 **LEM** Let $\mathbb{X}$ be a topological space and $\langle x_n\rangle \in \mathbb{X}^\omega$ a convergent sequence with $x_n \to x_\infty \in \mathbb{X}$. Define the corresponding subspace $\mathbb{X}_0 := \{x_n \mid n \in \mathbb{N} \cup \{\infty\}\}$. Note $x_n \to x_\infty \in \mathbb{X}_0$ as well. Then the map $\rho_0 :\subseteq \mathbf{C} \twoheadrightarrow \mathbb{X}_0$ defined by

$$\mathrm{dom}(\rho_0) := \mathbf{C}_0 \quad \text{with} \quad \forall n \in \mathbb{N} \cup \{\infty\}.\ \rho_0(\alpha_n) := x_n$$

is a continuous spanned representation.

*P–f* By inspection, $\rho_0$ is a representation of $\mathbb{X}_0$ with spanning forest $\mathcal{F} := \{\downarrow w \mid \exists \alpha \in \mathbf{C}_0.\ w \in \alpha\} \subseteq \mathbf{C}_{\mathrm{fin}}$. Let $V \subseteq \mathbb{X}_0$ be an open subset. Suppose $\alpha_n \in \rho_0^{-1}(V)$. Note $\{\alpha_n\} = \uparrow 0^n 1 \cap \mathbf{C}_0$ is open in $\mathbf{C}_0$. Thus $\alpha_n$ is an interior point of $\rho_0^{-1}(V)$. Now suppose $\alpha_\infty \in \rho_0^{-1}(V)$. Given that $x_n \to x_\infty \in \mathbb{X}_0$, $\exists N$ such that $\forall n \geq N.\ \rho_0(\alpha_n) = x_n \in V$. Note $\{\alpha_n \mid n \geq N\} \cup \{\alpha_\infty\} = \uparrow 0^N \cap \mathbf{C}_0$ is open in $\mathbf{C}_0$. Thus $\alpha_\infty$ is an interior point of $\rho_0^{-1}(V)$. Therefore $\rho_0^{-1}(V)$ is open in $\mathbf{C}_0$, i.e. $\rho_0$ is a continuous spanned representation. ∎

2.8 **THM** Let $\mathbf{X} \xrightarrow{\rho} \mathbb{X}$ and $\mathbf{Y} \xrightarrow{\varphi} \mathbb{Y}$ be admissible representations. Then a map $f : \mathbb{X} \to \mathbb{Y}$ is stably realizable via $\rho$ to $\varphi$ iff $f$ is sequentially continuous.

[1] They required countable token sets in their proof but the theory generalizes without relying on the categorical equivalence.

*P–f* (⇒) Suppose $\langle x_n \rangle \in \mathbb{X}^\omega$ is a convergent sequence with $x_n \to x_\infty \in \mathbb{X}$. Following the construction in Lemma 2.7, define the representation $\mathbf{C} \xrightarrow{\rho_0} \mathbb{X}_0$. Then the following diagram commutes by admissibilty.

$$\begin{array}{ccccc} \mathbf{C} & \overset{G}{\dashrightarrow} & \mathbf{X} & \overset{F}{\dashrightarrow} & \mathbf{Y} \\ \rho_0 \downarrow & & \rho \downarrow & & \downarrow \varphi \\ \mathbb{X}_0 & \overset{i}{\hookrightarrow} & \mathbb{X} & \overset{f}{\longrightarrow} & \mathbb{Y} \end{array}$$

where $G$ and $F$ are stable maps that realize $i$ and $f$ respectively. By construction

$$\forall n \in \mathbb{N} \cup \{\infty\}.\ f(x_n) = (f \circ i \circ \rho_0)(\alpha_n) = (\varphi \circ F \circ G)(\alpha_n).$$

Recall $\alpha_n \to \alpha_\infty \in \mathbf{C}$. By continuity, $(\varphi \circ F \circ G)(\alpha_n) \to (\varphi \circ F \circ G)(\alpha_\infty) \in \mathbb{Y}$ and thus $f(x_n) \to f(x_\infty) \in \mathbb{Y}$. Therefore $f$ is sequentially continuous.

(⇐) TBD. ■

### 2.4 Representation of Banach Spaces

Matsumoto and Terui sought to represent the real line with a coherence space. They achieved this via Cauchy sequences. We generalize this construction to Banach spaces, i.e. complete normed vector spaces (over $\mathbb{R}$ or $\mathbb{C}$).

**DEF** Let $\mathbb{X}$ be a Banach space with norm $\|\cdot\|$ and dense subset $D \subseteq \mathbb{X}$. The *dyadic space* of $\mathbb{X}$ (wrt $D$) is the coherence space $\mathbf{X}$ where $|\mathbf{X}| = \mathbb{N} \times D$ and $(m, x) \frown (n, x')$ iff $m \neq n$ and $\|x - x'\| \leq 2^{-m} + 2^{-n}$.

Intuitively, the dyadic space is a partialized space of Cauchy sequences over $D$. Let's make this idea precise. Abusing notation, we identify $z \coloneqq (n, x) \in |\mathbf{X}|$ with $x \in D$ and define $\mathrm{idx}(z) \coloneqq n$. That is, we treat tokens of $\mathbf{X}$ as vectors in $D$ with extra structure. For convenience, define $\forall k \in \mathbb{N}$ and $z \in |\mathbf{X}|$

$$|\mathbf{X}|^{(k)} \coloneqq \{(n, x) \in |\mathbf{X}| \mid n = k\} \quad \text{and} \quad [z] \coloneqq \left\{x \in \mathbb{X} \mid \|z - x\| \leq 2^{-\mathrm{idx}(z)}\right\}.$$

Note each $|\mathbf{X}|^{(k)}$ partitions $|\mathbf{X}|$ into cocliques. Thus each $\alpha \in \mathbf{X}$ contains at most one $z_k \in |\mathbf{X}|^{(k)}$. Also note $[z] \cap [z'] \neq \emptyset$ iff $\|z - z'\| \leq 2^{-\mathrm{idx}(z)} + 2^{-\mathrm{idx}(z')}$. Thus each $\alpha \in \mathbf{X}$ can be extended to contain at least one $z_k \in |\mathbf{X}|^{(k)}$ given that $D$ is dense. Therefore each $\alpha \in \mathbf{X}_{\max}$ corresponds to a (rapidly-converging) Cauchy sequence $\langle z_k \rangle$ such that

$$\forall \varepsilon > 0.\, \exists N \in \mathbb{N}.\, \forall m, n > N.\ \|z_m - z_n\| \leq 2^{-m} + 2^{-n} \leq 2^{-N} < \varepsilon$$

where $\forall k \in \mathbb{N}.\, \exists!\, z_k \in \alpha \cap |\mathbf{X}|^{(k)}$. Since $\mathbb{X}$ is complete, $\forall \alpha \in \mathbf{X}_{\max}.\ \lim_{n\to\infty} z_n \in \mathbb{X}$. Since $D$ is dense, every $x \in \mathbb{X}$ is the limit of some (rapidly-converging) Cauchy sequence $\langle z_n \rangle \in D^\omega$. This induces a natural representation of $\mathbb{X}$.

**DEF** Let $\mathbb{X}$ be a Banach space with dyadic space $\mathbf{X}$. The *Cauchy representation* of $\mathbb{X}$ (over $\mathbf{X}$) is $\mathbf{X} \xrightarrow{\rho_{\mathbf{X}}} \mathbb{X}$ defined by $\mathrm{dom}(\rho_{\mathbf{X}}) \coloneqq \mathbf{X}_{\max}$ with $\rho_{\mathbf{X}}(\alpha) \coloneqq \lim_{n\to\infty} z_n \in \mathbb{X}$ where $\langle z_n \rangle$ is the corresponding Cauchy sequence of $\alpha$.

For convenience, we denote $\rho_{\mathbf{X}}(\alpha)$ as $\alpha^\star$ where $\alpha \in \mathbf{X}_{\max}$.

2.9 **LEM** Let $\mathbf{X} \xrightarrow{\rho_{\mathbf{X}}} \mathbb{X}$ be a Cauchy representation. Then $\rho_{\mathbf{X}} : \mathbf{X}_{\max} \to \mathbb{X}$ is continuous.

*P–f* TBD. ■