Density of Critical Points and Integer Partitioning

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Abstract

Analysis and numerical experiments on the density of critical points in finite gap solutions for the NLS equation. The density of critical points is posed as the distribution of a partition problem.

Contents

1 Problem Statement

Consider a sample space $\Omega \coloneqq \{-1, 0, 1\}$ with probability mass function $\rho_\bullet : \Omega \to [0, 1]$ such that $\rho_{-1} = \rho_1$. We aim to analyze the distribution of the discrete random variable

$$\mathbf{s}_n \coloneqq \sum_{k=1}^{n} \frac{k}{n}\mathbf{w}_k \quad \text{as} \quad n \to \infty \quad \text{where} \quad \mathbf{w}_k \in \Omega \tag{1}$$

since this coincides with the density of critical points in finite gap solutions for the fNLS equation. Note the only possible values of $\mathbf{s}_n$ are of the form m/n where $m \in \mathbb{Z}$ and $|m| \leq n(n+1)/2$. Observe

$$\mathbf{s}_n = \sum_{k=1}^{n} \frac{k}{n}\mathbf{w}_k = \frac{m}{n} \quad \Longleftrightarrow \quad \mathbf{r}_n \coloneqq \sum_{k=1}^{n} k\mathbf{w}_k = m \tag{2}$$

and hence the distribution of $\mathbf{s}_n$ is equivalent to that of a probabilistic generalization of an integer partition problem $\mathbf{r}_n$.

2 Recurrence Relations

Fortunately, we can calculate the distribution of $\mathbf{r}_n$ efficiently by observing that the last term of $\mathbf{r}_{n+1}$ can be expanded to three cases, each reducing to $\mathbf{r}_n$.

Theorem 1 *Define a sequence of functions $\{R_n : \mathbb{Z} \to \mathbb{R}\}_{n=0}^{\infty}$ recursively by*

Base: $R_0(0) := 1$ *and* $R_0(m) := 0$ *for all* $m \in \mathbb{Z} \setminus \{0\}$, *and*

Recursion: $R_{n+1}(m) := \sum_{w\in\Omega} \rho_w \cdot R_n(m - w(n+1))$ *for all* $n \in \mathbb{N}_0$ *and* $m \in \mathbb{Z}$.

Then $P(\mathbf{s}_n = {}^m\!/_n) = P(\mathbf{r}_n = m) = R_n(m)$ *for all* $n \in \mathbb{N}$ *and* $m \in \mathbb{Z}$.

By Theorem 1, R_n is a discrete probability distribution. Moreover by definition of $\mathbf{r}_n$, it follows that R_n is even with compact support over $[{-n(n+1)}/2, {n(n+1)}/2] \cap \mathbb{Z}$. Given that $\rho_{-1} = \rho_1$, the recursive definition can be simplified to

$$R_{n+1}(m) = \rho_0 \cdot R_n(m) + \rho_1 \cdot [R_n(m-n-1) + R_n(m+n+1)]\,. \tag{3}$$

Using the theory of formal Laurent series, we can reframe R_n as a generating function. Define for each $n \in \mathbb{N}_0$,

$$f_n(z) := \sum_{m\in\mathbb{Z}} R_n(m) z^m. \tag{4}$$

Applying Eq. (3) to f_{n+1} then expanding and index shifting, we obtain the recurrence relation

$$f_{n+1}(z) = \left[\rho_0 + \rho_1 \left(z^{-n-1} + z^{n+1}\right)\right] \cdot f_n(z) \quad \text{for all} \quad n \in \mathbb{N}_0 \tag{5}$$

with base case $f_0 \equiv 1$. In fact, this recurrence relation over functions admits an explicit solution.

Theorem 2 *Let* $n \in \mathbb{N}_0$. *Then*

$$f_n(z) = \prod_{k=1}^{n} \left[\rho_0 + \rho_1 \left(z^{-k} + z^k\right)\right] \tag{6}$$

with the convention $\prod_{k=1}^{0}[\dots] := 1$.

3 Saddle-Point Asymptotics

From here on we will consider the special case $\rho_0 = 1/2$ (i.e. $\rho_1 = 1/4$). Through some algebra we obtain

$$f_n(z) = \frac{1}{4^n} \prod_{k=1}^{n} \left[z^{-k} + 2 + z^k\right] = \frac{1}{2^{2n+1}} \prod_{k=-n}^{n} \left[1 + z^k\right] = \frac{1}{2^{2n+1}} \sum_{m\in\mathbb{Z}} A_n(m) z^m \tag{7}$$

where $A_n(m)$ is the number of ways m can be partitioned into $\sum_{k=-n}^{n} k\varepsilon_k$ with $\varepsilon_k \in \{0,1\}$. R.C. Entringer proved a surprising result in his paper [Ent68] on the asymptotics of $A_n(m)$.

Theorem 3 *Fix* $t \in \mathbb{N}_0$ *and relate* $m = tn$. *Then*

$$A_n(tn) \sim (3/\pi)^{1/2}\, 2^{2n+1} n^{-3/2} \quad as \quad n \to \infty. \tag{8}$$

Since $R_n(m) = 1/2^{2n+1} A_n(m)$, *it follows that*

$$R_n(tn) \sim (3/\pi)^{1/2} n^{-3/2} \quad as \quad n \to \infty. \tag{9}$$

That said, we aim to study the asymptotic decay of the tail of the distribution. Hence we need to relate m with n so that m grows comparable to the support of R_n. Given that the endpoint of the support is $n(n+1)/2$, we focus on the relation

$$m = \alpha n^2 \quad \text{where} \quad \alpha \in [0, 1/2]. \tag{10}$$

3.1 Residue Theory

Since f_n is a finite product of Laurent polynomials, it is also a Laurent polynomial and hence an object of study in complex analysis. Indeed, we can recover the coefficient $R_n(m)$ of z^m for each $m \in \mathbb{Z}$ using Cauchy's integral formula.

Theorem 4 *Let $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$. Then*

$$R_n(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imt} \prod_{k=1}^{n} \cos^2\left(\frac{kt}{2}\right) \, dt. \tag{11}$$

Note that Eq. (11) is an oscillatory integral with parameter $m \gg n$. The integrand is especially peculiar in that it rapidly vanishes outside $[-\pi/n, \pi/n]$ as $n \to \infty$.

Conjecture 5 *Let $m \gg n$. Then the leading asymptotic behavior of R_n is given by*

$$R_n(m) \sim \frac{1}{2\pi} \int_{-\pi/n}^{\pi/n} e^{imt} \prod_{k=1}^{n} \cos^2\left(\frac{kt}{2}\right) \, dt \quad \text{as} \quad n \to \infty. \tag{12}$$

Supposing Conjecture 5 true, we can substitute Eq. (10) and $\xi := nt$ into Eq. (12) to normalize the integration bounds.

$$R_n(\alpha n^2) \sim \frac{1}{2\pi n} \int_{-\pi}^{\pi} e^{i\alpha n\xi} \exp\left\{\sum_{k=1}^{n} \ln\cos^2\left(\frac{k\xi}{2n}\right)\right\} d\xi \quad \text{as} \quad n \to \infty. \tag{13}$$

Note the sum in Eq. (13) is nonpositive and concave down on $\xi \in [-\pi, \pi]$. Consequently, the exponential contributes far less to the integral past its stationary point at $\xi = 0$.

Conjecture 6 *Suppose Conjecture 5 is true. Then the leading asymptotic behavior of $R_n(\alpha n^2)$, where $\alpha \in [0, 1/2]$, is given by*

$$R_n(\alpha n^2) \sim \frac{1}{2\pi n} \int_{-\pi}^{\pi} e^{i\alpha n\xi} \exp\left\{\frac{n}{\xi} \int_0^{\xi} \ln\cos^2\frac{x}{2} \, dx\right\} d\xi \quad \text{as} \quad n \to \infty. \tag{14}$$

Note the nested integral comes from the observation that the sum in Eq. (13) takes the form of a right Riemann sum as $n \to \infty$ if we scale by $1/n$.

With some renaming, we can present Eq. (14) more succinctly.

$$R_n(\alpha n^2) \sim \frac{1}{2\pi n} \int_{-\pi}^{\pi} e^{n(\psi(z)+i\alpha z)} \, dz \quad \text{where} \quad \psi(z) := \frac{1}{z}\int_0^{z} \ln\cos^2\frac{x}{2} \, dx. \tag{15}$$

3.2 Digression on ψ

Using Wolfram Alpha, we can express ψ in analytic form.

$$\psi(z) = \frac{1}{z}\left[2i\,\mathrm{Li}_2(-e^{ix}) + \frac{ix^2}{2} - 2x\ln(1+e^{ix}) + x\ln\cos^2\frac{x}{2}\right]_{x=0}^{z} \tag{16}$$

where Li_n is the polylogarithm, also known as Jonquière's function, defined by

$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}. \tag{17}$$

Note $\mathrm{Li}_2(-1) = -\pi^2/12$. The polylogarithm has some especially nice properties including the fact

$$\frac{d}{dz}\mathrm{Li}_n(z) = \frac{1}{z}\mathrm{Li}_{n-1}(z). \tag{18}$$

We can visualize the phase function in Eq. (15) over $\mathbb{C}$. Consider the case $\alpha = 1/4$. The following is the real (left) and imaginary (right) graphs of $\psi(z) + i\alpha z$ as well as its projections onto $\mathfrak{Re}\ z$ (Fig. 1) and $\mathfrak{Im}\ z$ (Fig. 2) respectively.

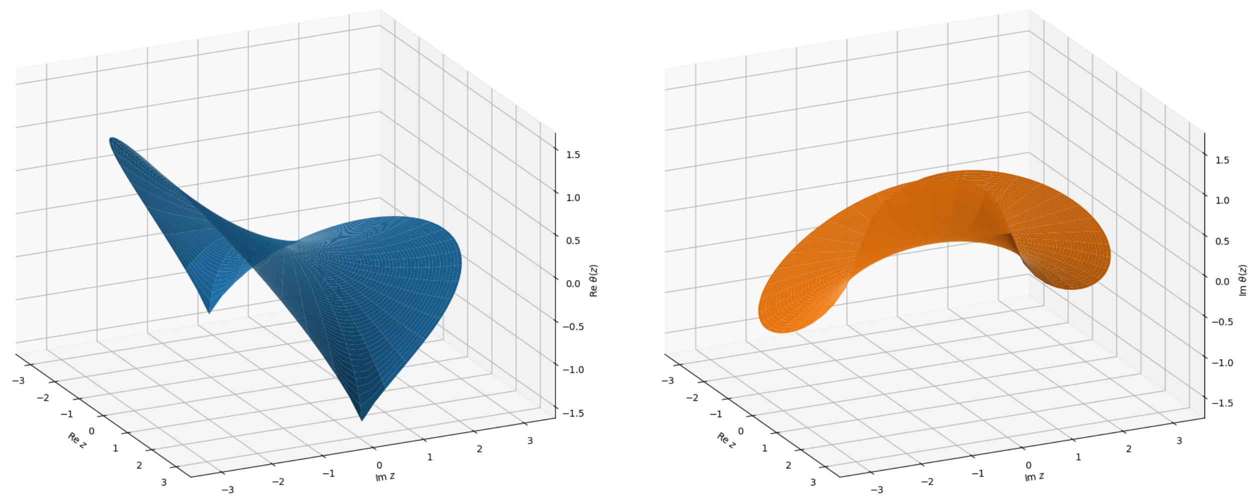

Figure 1: Real and imaginary graphs of $\psi(z) + i\alpha z$.

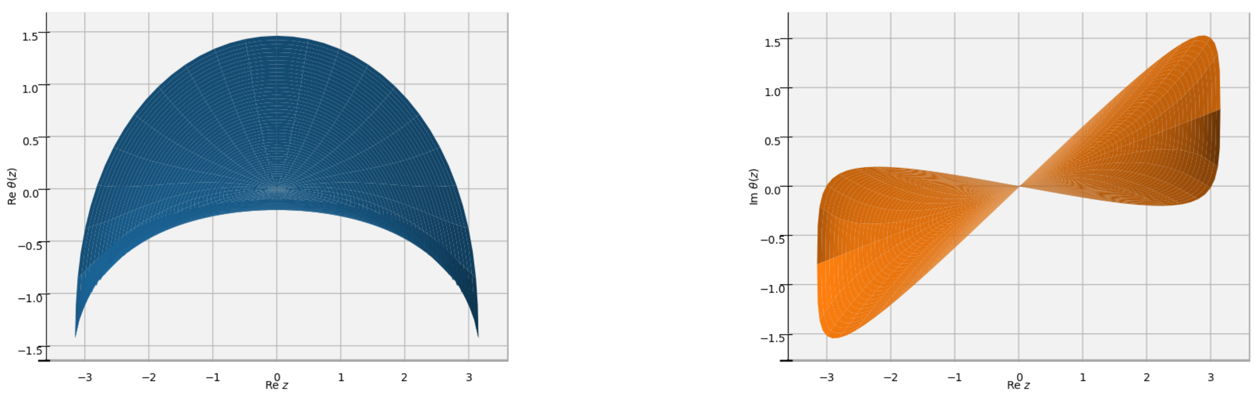

Figure 2: Real and imaginary graphs of $\psi(z) + i\alpha z$ projected onto $\mathfrak{Re}\ z$.

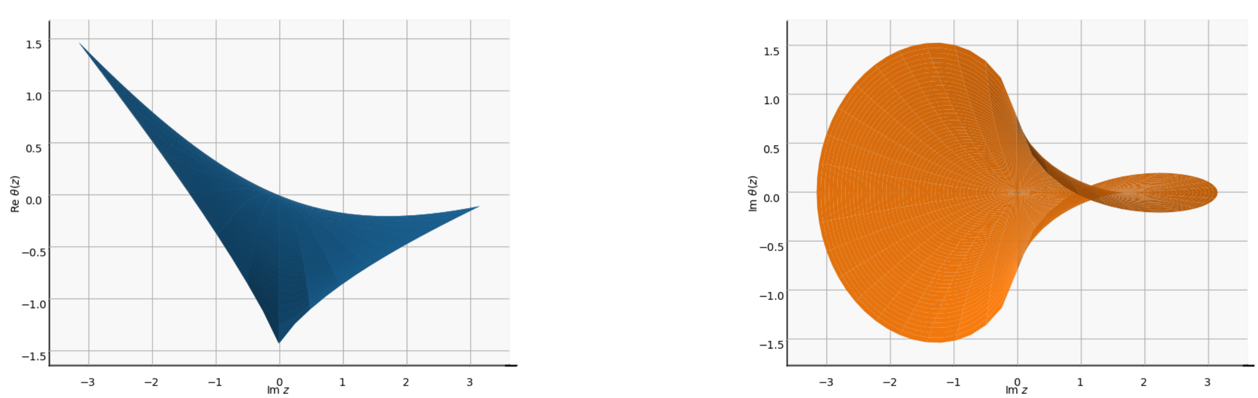

Figure 3: Real and imaginary graphs of $\psi(z) + i\alpha z$ projected onto $\mathfrak{Im}\ z$.

3.3 Method of Steepest Descent

Recall the following theorem for approximating oscillatory integrals.

Theorem 7 (Method of Steepest Descent) *Consider the oscillatory integral*

$$I(\lambda) = \int_a^b g(z) e^{\lambda f(z)}\, dz \quad as \quad \lambda \to \infty$$

where f and g are analytic functions. Suppose we can deform the contour of integration to pass through a single stationary point z_0 of f. Then the oscillatory integral is given by the asymptotic

$$I(\lambda) \sim g(z_0) e^{\lambda f(z_0)} e^{i\phi} \left(\frac{2\pi}{\lambda |f''(z_0)|} \right)^{1/2} \quad as \quad \lambda \to \infty \quad where \quad \phi = \frac{\pi - \arg f''(z_0)}{2}.$$

Continuing with Eq. (15), define $\theta(z) \coloneqq \psi(z) + i\alpha z$. The corresponding derivatives are

$$z\theta'(z) + \theta(z) = \partial_z [z\theta(z)] = \partial_z \left[\int_0^z \ln \cos^2 \frac{x}{2}\, dx + i\alpha z^2 \right] = \ln \cos^2 \frac{z}{2} + 2i\alpha z \tag{19}$$

$$z\theta''(z) + 2\theta'(z) = \partial_z^2 [z\theta(z)] = \partial_z \left[\ln \cos^2 \frac{z}{2} + 2i\alpha z \right] = -\tan \frac{z}{2} + 2i\alpha. \tag{20}$$

Based on the numerics in Section 3.2, we have the following.

Conjecture 8 *For all $\alpha \in [0, 1/2]$, $\theta(z)$ has an isolated stationary point on the nonnegative imaginary axis of z. That is, there exists $x_0 \in [0, \infty)$ such that $\theta'(ix_0) = 0$ and $\theta''(ix_0) < 0$.*

Evaluating at the stationary point asserted in Conjecture 8, we have

$$\theta(ix_0) = \ln \cosh^2 \frac{x_0}{2} - 2\alpha x_0, \tag{21}$$

$$\theta''(ix_0) = \frac{1}{x_0} \left(2\alpha - \tanh \frac{x_0}{2} \right). \tag{22}$$

Therefore, we can apply the saddle point method to Eq. (15) and obtain the asymptotic

$$R_n(\alpha n^2) \sim n^{-3/2} e^{n\theta(ix_0)} e^{i\phi} \frac{1}{\sqrt{2\pi |\theta''(ix_0)|}} \quad \text{where} \quad \phi = \frac{\pi - \arg \theta''(ix_0)}{2} = 0 \tag{23}$$

$$\sim n^{-3/2} \exp\left\{ n \left(\ln \cosh^2 \frac{x_0}{2} - 2\alpha x_0 \right) \right\} \left(\frac{2\pi}{x_0} \left| 2\alpha - \tanh \frac{x_0}{2} \right| \right)^{-1/2} \quad \text{as} \quad n \to \infty. \tag{24}$$

4 Alpha Asymptotics

It remains to be seen the relationship between the saddle point $z = ix_0$ and the proportionality constant α. By definition of stationary point,

$$\partial_z|_{z=ix_0} [\psi(z) + i\alpha z] = \psi'(ix_0) + i\alpha = 0 \quad \Longrightarrow \quad \alpha = i\psi'(ix_0). \tag{25}$$

Expanding Eq. (25) and taking its Taylor expansion, we obtain the formula

$$\alpha = \frac{1}{x_0} \left(\ln \cosh^2 \frac{x_0}{2} - \frac{1}{x_0} \int_0^{x_0} \ln \cosh^2 \frac{x}{2}\, dx \right) = \frac{1}{6} x_0 - \frac{1}{120} x_0^3 + \mathcal{O}(x_0^5). \tag{26}$$

Unfortunately, there is no simple inversion of Eq. (26) that solves x_0 in terms of α. Instead, we will focus on asymptotics when alpha approaches its extreme values.

4.1 Small Scale Asymptotics

Recall the following theorem for inverting analytic functions.

Theorem 9 (Lagrange Inversion Theorem) *Suppose $z = f(w)$ where f is analytic at $w = w_0$ and $f'(w_0) \neq 0$. Then f has an inverse g analytic at $z = f(w_0)$ given by the power series*

$$g(z) = w_0 + \sum_{k=1}^{\infty} \frac{g_k}{k!}(z - f(w_0))^k \quad \textit{where} \quad g_k = \lim_{w \to w_0} \partial_w^{k-1} \left[\frac{w - w_0}{f(w) - f(w_0)} \right]^k,$$

i.e. $g(z) = w$ in a neighborhood of $z = f(w_0)$.

By inspection of Eq. (26), it follows $i\psi'(iz)$ is analytic at $z = 0$ with $\partial_z|_{z=0} [i\psi'(iz)] = 1/6 \neq 0$. Thus, we can apply the Lagrange Inversion Theorem to obtain a formula for x_0 for sufficiently small α.

$$x_0 = \sum_{k=1}^{\infty} \frac{g_k}{k!} \alpha^k \quad \text{where} \quad g_k = \lim_{x \to 0} \partial_x^{k-1} \left[\frac{x}{i\psi'(ix)} \right]^k. \tag{27}$$

Calculating the first couple of terms of the series, we obtain the series

$$x_0 = 6\alpha + \frac{324}{5}\alpha^3 + \frac{128304}{35}\alpha^5 + \mathcal{O}(\alpha^7) \quad \text{where} \quad \alpha \to 0. \tag{28}$$

Substituting into Eqs. (21) and (22) and expanding coefficients, we obtain estimates

$$\theta(ix_0) = \ln\cosh^2 \frac{x_0}{2} - 2\alpha x_0 = -3\alpha^2 + \frac{513}{10}\alpha^4 + \mathcal{O}(\alpha^6) \sim -3\alpha^2, \tag{29}$$

$$\theta''(ix_0) = \frac{1}{x_0}\left(2\alpha - \tanh\frac{x_0}{2}\right) = -\frac{1}{6} - \frac{21}{10}\alpha^2 - \frac{24111}{175}\alpha^4 + \mathcal{O}(\alpha^6) \sim -\left(\frac{1}{6} + \frac{21}{10}\alpha^2\right). \tag{30}$$

as $\alpha \to 0$. Hence Eq. (24) reduces to a small scale asymptotic,

$$R_n(\alpha n^2) \sim n^{-3/2} \exp\left\{-3n\alpha^2\right\} \left(\frac{\pi}{3} + \frac{21\pi}{5}\alpha^2\right)^{-1/2} \quad \text{as} \quad n \to \infty. \tag{31}$$

In fact when $\alpha = 0$, the quadratic relation $m = \alpha n^2$ collapses to a linear relation because $m = 0n^2 = 0n$. Thus, we expect the asymptotic given by Eq. (31) to be equivalent to Eq. (9) in the limiting case $\alpha = 0$. Indeed

$$R_n(0n^2) \sim n^{-3/2} \exp\{0\} (\pi/3 + 0)^{-1/2} = (3/\pi)^{1/2} n^{-3/2} \sim R_n(0n). \tag{32}$$

4.2 Large Scale Asymptotics

Rearranging Eq. (26), we obtain the expression

$$\frac{\alpha}{2}x_0^2 = x_0 \ln\cosh\frac{x_0}{2} - \int_0^{x_0} \ln\cosh\frac{t}{2}\, dt. \tag{33}$$

Expanding definitions on the right, we obtain

$$\frac{\alpha}{2}x_0^2 = x_0\left(\frac{x_0}{2} + \ln(1 + e^{-x_0}) - \ln 2\right) - \int_0^{x_0} \left(\frac{t}{2} + \ln(1 + e^{-t}) - \ln 2\right) dt \tag{34}$$

$$= \frac{1}{4}x_0^2 + x_0 \ln(1 + e^{-x_0}) - \int_0^{x_0} \ln(1 + e^{-t})\, dt. \tag{35}$$

Intuitively, $x_0 \to \infty$ as $\alpha \to 1/2$. We can make this precise by solving the above equations as follows.

$$\frac{1 - 2\alpha}{4}x_0^2 = \int_0^{x_0} \ln(1 + e^{-t})\, dt - x_0 \ln(1 + e^{-x_0}) \tag{36}$$

$$x_0 = \frac{2}{\sqrt{1 - 2\alpha}} \left[\int_0^{x_0} \ln(1 + e^{-t})\, dt - x_0 \ln(1 + e^{-x_0})\right]^{1/2}. \tag{37}$$

Taking $\alpha \to 1/2$, the integral converges and the linear-log expression vanishes. Thus we obtain the rough estimate

$$x_0 \sim \frac{2}{\sqrt{1-2\alpha}}\left[\frac{\pi^2}{12} - x_0 e^{-x_0}\right]^{1/2} \sim \frac{\pi}{\sqrt{3(1-2\alpha)}} \quad \text{as} \quad \alpha \to \frac{1}{2}. \tag{38}$$

Substituting into Eqs. (21) and (22) and expanding coefficients, we obtain estimates

$$\theta(ix_0) = \ln\cosh^2\frac{x_0}{2} - 2\alpha x_0 \sim (1-2\alpha)x_0 + 2e^{-x_0} - \ln 4 \sim \frac{\pi}{\sqrt{3}}(1-2\alpha)^{1/2} - \ln 4 \tag{39}$$

$$\theta''(ix_0) = \frac{1}{x_0}\left(2\alpha - \tanh\frac{x_0}{2}\right) \sim \frac{1}{x_0}(2\alpha - 1 + 2e^{-x_0}) \sim -\frac{\sqrt{3}}{\pi}(1-2\alpha)^{3/2} \tag{40}$$

as $\alpha \to 1/2$. Hence Eq. (24) reduces to a large scale asymptotic,

$$R_n(\alpha n^2) \sim n^{-3/2}4^{-n}\exp\left\{n\pi\sqrt{\frac{1-2\alpha}{3}}\right\}\left(12(1-2\alpha)^3\right)^{-1/4} \quad \text{as} \quad n \to \infty. \tag{41}$$

5 Validating Conjectures

Recall that the density of critical points is given exactly by the formula

$$R_n(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imt}\prod_{k=1}^{n}\cos^2\left(\frac{kt}{2}\right)\,dt. \tag{42}$$

After applying a change of variables $\xi \coloneqq nt$ and substituting $m = \alpha n^2$, we obtain

$$R_n(\alpha n^2) = \frac{1}{2\pi n}\int_{-n\pi}^{n\pi} e^{i\alpha n\xi}\exp\left\{\sum_{k=1}^{n}\ln\cos^2\left(\frac{k\xi}{2n}\right)\right\}\,dt. \tag{43}$$

In order to apply the saddle point method, we jumped from

$$\sum_{k=1}^{n}\ln\cos^2\left(\frac{k\xi}{2n}\right) \quad \text{to} \quad \frac{n}{\xi}\int_0^{\xi}\ln\cos^2\frac{x}{2}\,dx. \tag{44}$$

5.1 Amplitude Dependence on Large Parameter

One way to justify this transition is by the following. Observe

$$R_n(\alpha n^2) = \frac{1}{2\pi n}\int_{-n\pi}^{n\pi} A(n,\xi)e^{i\alpha n\xi}\exp\left\{\frac{n}{\xi}\int_0^{\xi}\ln\cos^2\frac{x}{2}\,dx\right\}\,dt. \tag{45}$$

$$\text{where} \quad A(n,\xi) = \exp\left\{n\cdot\left(\frac{1}{n}\sum_{k=1}^{n}\ln\cos^2\left(\frac{k\xi}{2n}\right) + i\alpha\xi - \frac{1}{\xi}\int_0^{\xi}\ln\cos^2\frac{x}{2}\,dx - i\alpha\xi\right)\right\}. \tag{46}$$

If we can demonstrate that $A(n,\xi) = A_0(\xi) + A_1(\xi)B(n)$ with $B(n) = \mathcal{O}(1)$ as $n \to \infty$ and $\xi \to \xi_0$ (the phase's saddle point), then we can apply the saddle point method as usual by ignoring A. This is equivalent to showing that

$$\Delta(n,\alpha) = \frac{1}{n}\sum_{k=1}^{n}\ln\cos^2\left(\frac{k\xi}{2n}\right) + i\alpha\xi - \left(\frac{1}{\xi}\int_0^{\xi}\ln\cos^2\frac{x}{2}\,dx + i\alpha\xi\right) = \mathcal{O}\left(\frac{1}{n}\right) \quad \text{as} \quad n\to\infty \quad \text{and} \quad \xi \to \xi_0. \tag{47}$$

(**NEED TO VERIFY THE ABOVE CLAIMS**).

5.2 Numerical Experiments

We know that the phase's saddle point $\xi_0 = iy_0$ is located on the positive imaginary axis. I.e.

$$y_0 = \arg\min_{y\in\mathbb{R}} \left(\frac{1}{y} \int_0^y \ln\cosh^2 \frac{x}{2}\, dx - \alpha y \right) \tag{48}$$

We are thus looking to see the difference between the sum and the integral in Eq. (47) at the point ξ_0 where the integral takes its minimal value along the imaginary axis (Σ (at min$\int$) - min$\int$). That is,

$$\frac{1}{n} \sum_{k=1}^{n} \ln\cosh^2 \left(\frac{ky_0}{2n} \right) - \alpha y_0 - \frac{1}{y_0} \int_0^{y_0} \ln\cosh^2 \frac{x}{2}\, dx + \alpha y_0 \tag{49}$$

Similarly, we compare the minimal value of the sum with the minimal value of the integral along the imaginary axis (min Σ - min$\int$). That is,

$$\min_{y\in\mathbb{R}} \left(\frac{1}{n} \sum_{k=1}^{n} \ln\cosh^2 \left(\frac{ky}{2n} \right) - \alpha y \right) - \frac{1}{y_0} \int_0^{y_0} \ln\cosh^2 \frac{x}{2}\, dx + \alpha y_0 \tag{50}$$

Overall, we obtain the following calculations.

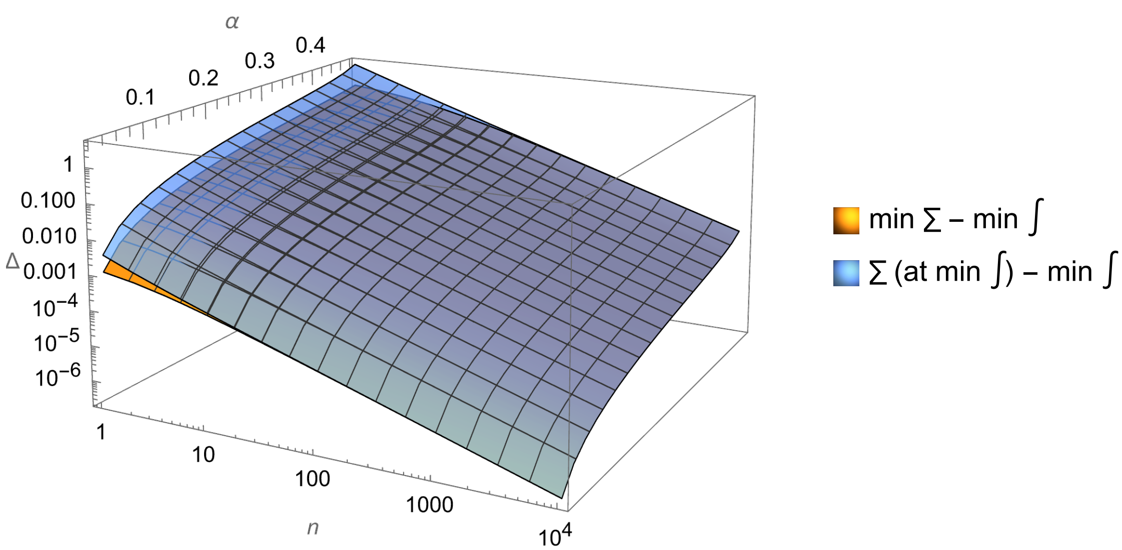

Figure 4: Difference (Δ) of sum (Σ) and integral ($\int$) where n and Δ are on a log scale.

Note that for different choices of $\alpha \in [0, 1/2]$, the difference decays in a linear fashion along the log-log plot. To see this clearer, consider the two extreme cases $\alpha = 1/42$ and $\alpha = 10/21$.

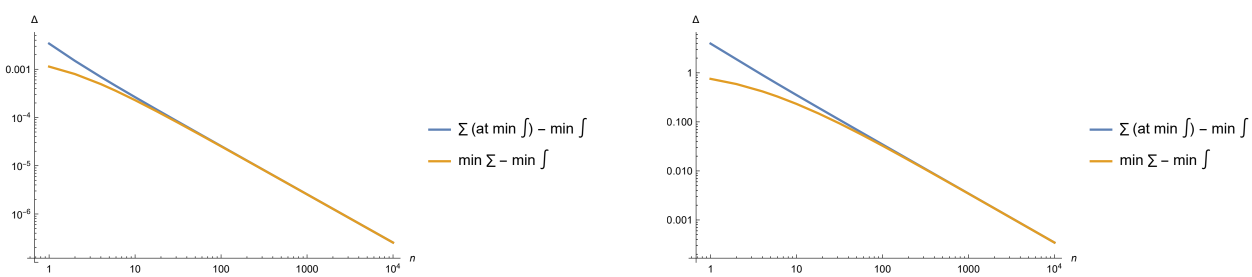

Figure 5: Difference of sum and integral for $\alpha = 1/42$ (left) and $\alpha = 10/21$ (right) where n and Δ are on a log scale.

As we can see much clearer, the difference decays linear in the log-log plot. This suggests that the difference satisfies a power law of the form

$$\Delta(n,\alpha) \sim \kappa(\alpha) n^{\rho(\alpha)} \quad \text{as} \quad n \to \infty \tag{51}$$

where κ is the coefficient and ρ is the power. Indeed, if we numerically approximate the coefficient and power for each α, we obtain

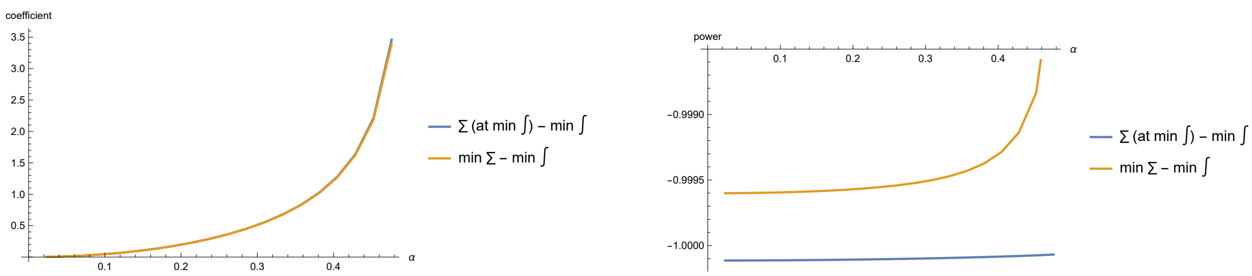

Figure 6: Approximation for the coefficient (left) and power (right) that defines the decay of the difference Δ for different $\alpha \in [0, 1/2]$.

For the difference of the sum and the integral at the saddle point of the phase Eq. (49) (Σ (at min$\int$) - min$\int$), we see that the power is slightly smaller than -1 (this may be due to approximation error) with coefficient starting at 0 when $\alpha = 0$ and appearing to grow to infinity as $\alpha \to 1/2$. This suggests that $\Delta = \mathcal{O}(1/n)$ about the phase's saddle point for $\alpha < 1/2$.

For the difference of the mnimum of the sum and the minimum of the integral Eq. (50) (min Σ - min$\int$), we see that the power is slightly larger than -1 (this may be due to approximation error) with coefficient matching that for the case of (Σ (at min$\int$) - min$\int$). This suggests that $\Delta = \Omega(1/n)$ about the phase's saddle point for $\alpha < 1/2$.

Combining the above two observations, we get numerical evidence that

$$\Delta(n,\alpha) = \Theta\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty \quad \text{and} \quad \xi \to \xi_0 \quad \text{for} \quad \alpha \in [0, 1/2) \tag{52}$$

which would imply that Eq. (43) satisfies the conditions to apply the saddle point method.

5.3 Addendum : Generalizing Saddle Point Method

See Daisuke Fujiwara's 2012 lecture notes at Gakushuin University titled "Stationary Phase Method, Feynman Path Integrals and Integration by Parts Formula" [Fuj12]. In section 2, it gives a stationary phase estimate of oscillatory integrals (Theorem 2.3) such that the amplitude and phase depend on the large parameter.

6 Digression on Riemann Surfaces

Recall the exact equation for $R_n(\alpha n^2)$ given by Eq. (11),

$$R_n(\alpha n^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\alpha n^2 t} \prod_{k=1}^{n} \cos^2\left(\frac{kt}{2}\right) dt \quad \text{as} \quad n \to \infty. \tag{53}$$

If we apply the subtitution $\xi := nt$ and convert the product to a sum of logs, we obtain the exact equation

$$R_n(\alpha n^2) = \frac{1}{2\pi n} \int_{-n\pi}^{n\pi} e^{i\alpha n \xi} \exp\left\{\sum_{k=1}^{n} \ln\cos^2\left(\frac{k\xi}{2n}\right)\right\} d\xi \quad \text{as} \quad n \to \infty. \tag{54}$$

Importantly, the sum of logs is independent of choice of branch cut for $\ln\cos^2$ since the multiple values are congruent modulo $4\pi i$ and hence contribute nothing to the exponential. As before, we frame the sum as a Riemann sum and take its limit to get the following conjecture

Conjecture 10 *Let $m \gg n$. Then the leading asymptotic behavior of $R_n(\alpha n^2)$, where $\alpha \in [0, 1/2]$, is given by*

$$R_n(\alpha n^2) \sim \frac{1}{2\pi n} \int_{-n\pi}^{n\pi} e^{n(\psi(z)+i\alpha z)}\, dz \quad \textit{where} \quad \psi(z) := \frac{1}{z}\int_0^z \ln\cos^2\frac{x}{2}\, dx. \tag{55}$$

As a point of contention, Conjecture 6 argues the sum can be subtituted by an integral since the sum converges when $\xi \in (-\pi, \pi)$. Here in contrast, the sum may diverge when $\xi \in (-n\pi, n\pi)$ for $n \geq 2$ especially since $\ln\cos^2 z/2$ is unbounded over such an interval and hence is not Riemann integrable. We continue in spite of this failure by assuming that we still get reasonable estimates when integrating over paths that avoid singularities.

To get from Eq. (55) to Eq. (14) (that is, slice the integral to the bounds $[-\pi, \pi]$), we could attempt to show that the contribution of the integrand over $[-n\pi, n\pi] \setminus [-\pi, \pi]$ is asymptotically smaller than the contribution over $[-\pi, \pi]$. As a way to make this precise, we could split Eq. (55) into a sum of integrals over $[-\pi, \pi]$, $[\pi, 3\pi]$, $[3\pi, 5\pi]$, ... and apply the method of steepest descent to each to get their corresponding asymptotic contribution.

However, in order to apply the method of steepest descent to the other sliced integrals (like the one over $[\pi, 3\pi]$), we require that ψ be analytic. To achieve this, we must pick suitable paths of integration that lie on the Riemann surface of $\ln\cos^2 z/2$.

6.1 Riemann Surface of $\ln\cos^2\frac{z}{2}$

Observe that the derivative of $\ln\cos^2\frac{z}{2}$ is the single-valued meromorphic function $-\tan z/2$ with poles at $z \in (2\mathbb{Z}+1)\pi$. Therefore for any z not at these poles,

$$\ln\cos^2\frac{z}{2} = \int_\gamma -\tan \tau/2\, d\tau \tag{56}$$

where $\gamma : [0, 1] \to \mathbb{C}$ is a path such that $\gamma(0) = 0$, $\gamma(1) = z$, and γ avoids the poles of the integrand.

Importantly, the domain of $-\tan z/2$ is a countably infinite punctured plane where the set of holes have no accumulation points. This means that the fundamental group of the domain is isomorphic to the free group with countably infinite generators, i.e. non-trivial. As a result, $\ln\cos^2 z/2$ is multivalued due to the existence of non-homotopic paths γ.

The generators of the fundamental group correspond to a single cycle around each pole. Since the residue of $-\tan z/2$ at each pole is 2, the line integral for any single cycle around a pole evaluates to $4\pi i$. Thus the multiple values of $\ln\cos^2 z/2$ are congruent modulo $4\pi i$.

Now suppose we aim to calculate $\ln\cos^2 z/2$. If we know its value at some point $z = x$, we can get its value at some neighboring point $z = x + \delta$ as follows. Let $\gamma = \gamma' * \gamma''$ be a path where γ' is a curve from $z = 0$ to $z = x$ such that $\ln\cos^2 x/2 = \int_{\gamma'} -\tan \tau/2\, d\tau$ and γ'' is a straight line from $z = x$ to $z = x + \delta$. Then

$$\ln\cos^2\frac{x+\delta}{2} = \int_\gamma -\tan\frac{\tau}{2}\, d\tau = \int_{\gamma'} -\tan\frac{\tau}{2}\, d\tau + \int_{\gamma''} -\tan\frac{\tau}{2}\, d\tau \tag{57}$$

$$= \ln\cos^2\frac{x}{2} + \int_0^\delta -\tan\frac{\tau+x}{2}\, d\tau. \tag{58}$$

Furthermore, $-\tan z/2$ having period 2π implies that

$$\ln\cos^2\frac{z+2\pi}{2} \in \ln\cos^2\frac{z}{2} + (C + 4\pi i)\mathbb{Z} \tag{59}$$

for some independent constant C. Indeed, $C = 2\pi i$. Since the multiple values are different only in their imaginary component, we can visualize the Riemann surface of $\ln\cos^2 z/2$ by graphing its imaginary component.

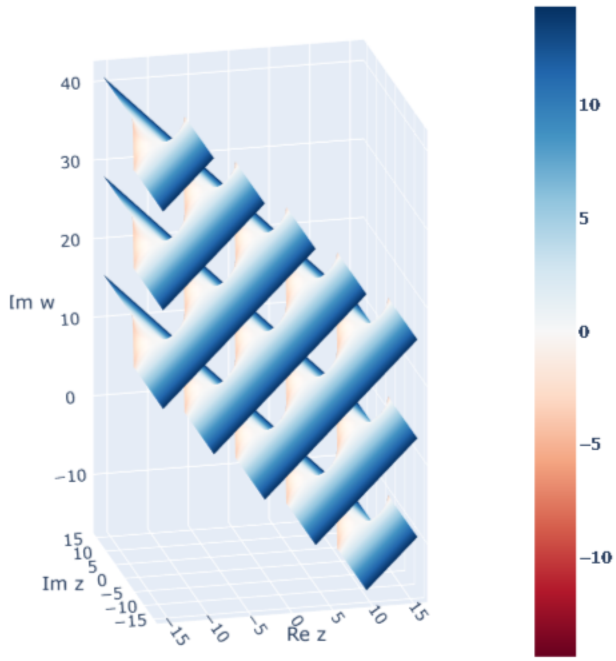

Figure 7: Imaginary graph of $w = \ln \cos^2 z/2$ with colorscale depicting Re w.

The above graph was computed by choosing a branch cut of the domain of $-\tan z/2$ and then adding the integral contribution of the fundamental group generators to compute other branches. In particular, the branch cut chosen is the following: at each pole, cut out a ray from that point to infinity in the negative imaginary direction. This ensures the resultant domain is simply connected.

Visually, if Im $z < 0$ then increasing Re z increases Im w. Conversely, if Im $z > 0$ then increasing Re z decreases Im w. Similarly, every counterclockwise loop around a pole increases Im w. Conversely, every clockwise loop around a pole decreases Im w. Moreover, Re w is symmetric along Im z. In particular, Re w is negative when approaching the poles. Otherwise, Re w is positive when approaching $\pm\infty i$.

6.2 Riemann Surface of $z\psi(z)$

Now that we have the Riemann surface of $\ln \cos^2 z/2$, we can proceed to calculate ψ. For simplicity of this subsection, we focus only on the integral over $\ln \cos^2 z/2$ (that is, $z\psi(z)$). Let

$$F(z) = z\psi(z) = \int_0^z \ln \cos^2 \frac{\tau}{2} \, d\tau. \tag{60}$$

As before, for any z not at the branch points of $\ln \cos^2 z/2$,

$$F(z) = \int_\gamma \ln \cos^2 \frac{\tau}{2} \, d\tau \tag{61}$$

where $\gamma : [0, 1] \to R$ is a path on the Riemann surface $R = \mathbb{C} \times \mathbb{Z}$ of $\ln \cos^2 z/2$ (with the second component representing the multiple of $4\pi i$) such that $\gamma(0) = (0, 0)$ and $\pi_1\gamma(1) = z$ (where π_1 is the projection of R onto $\mathbb{C}$).

Observe the Riemann surface R is homotopy equivalent to an infinite grid by essentially collapsing Im z (i.e. face the Re $z \times$ Im w-plane).

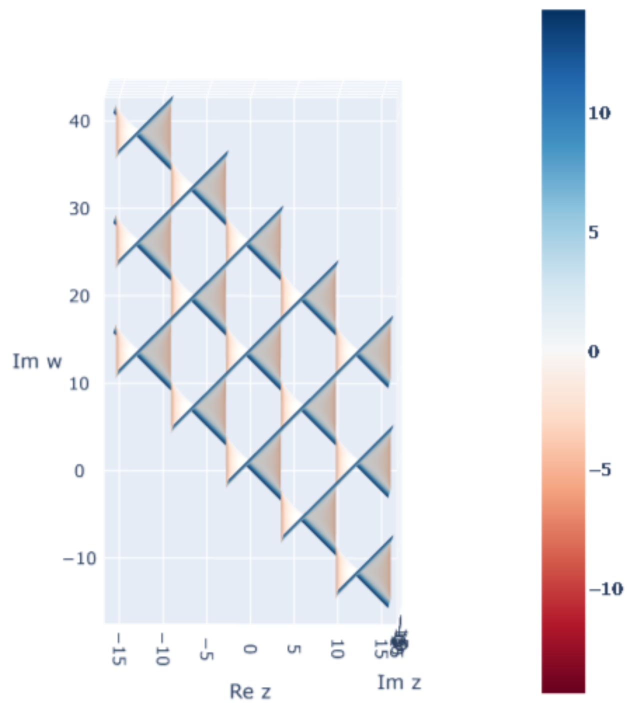

Figure 8: Imaginary graph of $w = \ln\cos^2 {}^{z}/_{2}$ facing Re $z \times$ Im w-plane with colorscale depicting Re w.

The infinite grid has fundamental group isomorphic to the free group with countably infinite generators, i.e. non-trivial. As a result, $F(z)$ may be multivalued due to the existence of non-homotopic paths γ.

The generators of the fundamental group correspond to a single cycle around each hole in the grid. Since the residue of $\ln\cos^2 {}^{z}/_{2}$ around each hole is 4π, the line integral for any single cycle around a hole evaluates to $8\pi^2 i$. However, this is not the only contribution to the multiple values of $F(z)$. Indeed, the value of $F(z)$ also depends on which level (the multiples of $4\pi i$) we land on the Riemann surface R.

Now suppose we aim to calculate $F(z)$. If we know its value at some point $z = x$, we can get its value at some neighboring point $z = x + \delta$ as follows. Let $\gamma = \gamma' * \gamma''$ be a path where γ' is a curve with $\gamma'(0) = (0, 0)$ and $\pi_1\gamma'(1) = x$ such that $F(z) = \int_{\gamma'} \ln\cos^2 {}^{\tau}/_{2}\, d\tau$. Moreover γ'' is a straight line with $\pi_1\gamma''$ going from $z = x$ to $z = x + \delta$. Then

$$F(x+\delta) = \int_\gamma \ln\cos^2\frac{\tau}{2}\,d\tau = \int_{\gamma'} \ln\cos^2\frac{\tau}{2}\,d\tau + \int_{\gamma''} \ln\cos^2\frac{\tau}{2}\,d\tau \tag{62}$$

$$= F(z) + \int_0^\delta \ln\cos^2\frac{x+\tau}{2}\,d\tau = F(z) + \int_0^\delta \left[\ln\cos^2\frac{x}{2} + \int_0^\tau -\tan\frac{y+x}{2}\,dy\right]\,d\tau \tag{63}$$

$$= F(z) + \delta\ln\cos^2\frac{x}{2} + \int_0^\delta\int_0^\tau -\tan\frac{y+x}{2}\,dy\,d\tau \tag{64}$$

$$= F(z) + \delta\ln\cos^2\frac{x}{2} + \int_0^\delta\int_y^\delta -\tan\frac{y+x}{2}\,d\tau\,dy \tag{65}$$

$$= F(z) + \delta\ln\cos^2\frac{x}{2} + \delta\int_0^\delta -\tan\frac{y+x}{2}\,dy + \int_0^\delta y\tan\frac{y+x}{2}\,dy \tag{66}$$

$$= F(z) + \delta\ln\cos^2\frac{x+\delta}{2} + \int_0^\delta \tau\tan\frac{\tau+x}{2}\,d\tau \tag{67}$$

Unlike the case of $\ln\cos^2\frac{z}{2}$, the multiple values of $F(z)$ differ in both their real and imaginary component making it slightly less obvious how one should visualize the surface. In the meantime, we can plot both components and color the surface based on the norm of the complex value to somewhat visualize how the surface is stitched.

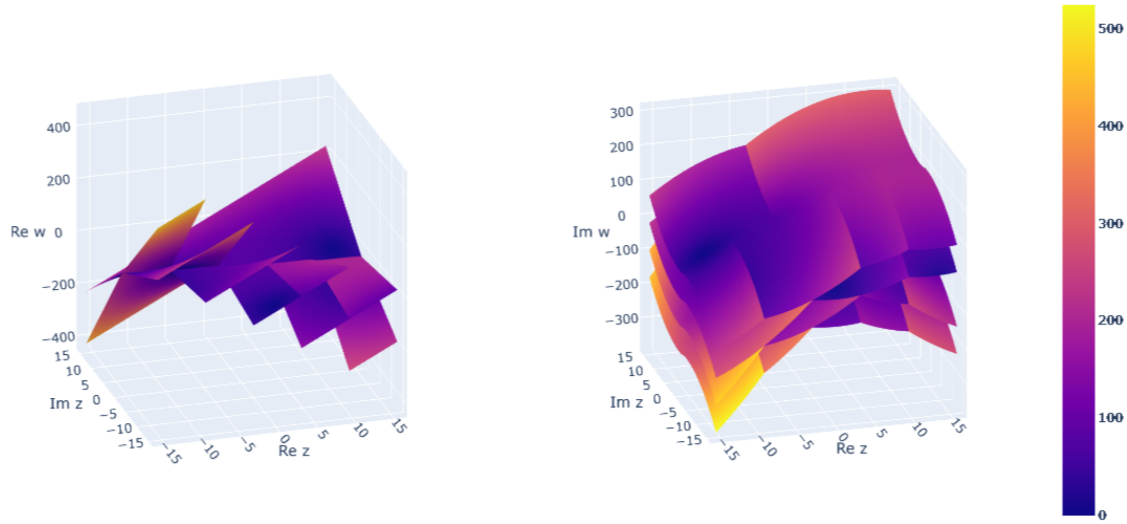

Figure 9: Real (left) and Imaginary (right) graphs of $w = F(z)$ with colorscale depicting $||w||$.

The above graph was computed by choosing a branch cut of the Riemann surface R and then adding the integral contribution of the fundamental group generators to compute other branches. In particular, the branch cut chosen is the following: at each branch point except for one (say at $z = -\pi$), cut out the ray extending from that point to infinity in the negative imaginary direction. The single uncut branch point (in this case at $z = -\pi$) ensures the resultant surface remains path connected. The other branch points with rays cut out ensures the resultant surface is simply connected. Equivalently, the chosen branch cut is the following spanning tree for the homotopy equivalent grid.

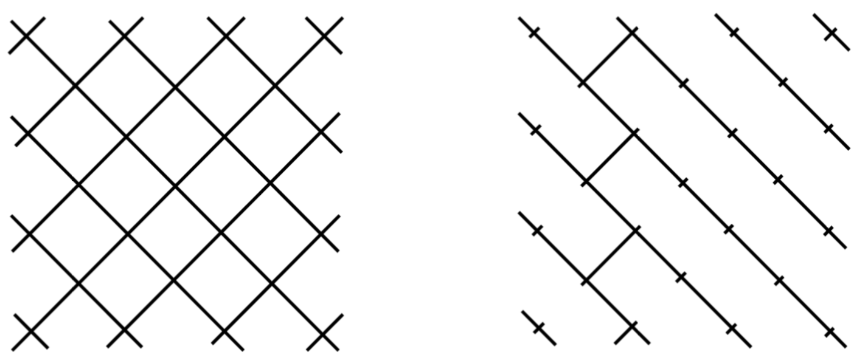

Figure 10: Homotopy equivalent infinite grid of $\ln \cos^2 {}^{z}\!/_{2}$ before cutting (left) and after cutting (right).

References

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